ACFD Assignment 3

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Poiseuille Benard Flow

Physical System

This problem represents a classical case of mixed convection flow in a horizontal rectangular channel heated from below. Poiseuille Benard flow is the result of the superposition of two convective sources - a horizontal pressure gradient giving rise to a forced flow characterized by Re and a vertical temperature gradient characterized by Ra.

The plates are of length $L = 10$ and have a spacing of $H = 1$ between them as shown in Fig. 1. Reynolds number is taken to be 10, Prandtl number as 0.7 and Rayleigh number as 10^4.The domain is divided into 100 elements in the y-direction and 400 elements in the streamwise direction (x-direction). A transient solution technique is adopted with time step of 10^{-3}.

Flow is assumed to be incompressible in nature and buoyancy force caused by density variation is taken into account using the Boussinesq approximation. The thermophysical properties are assumed to be invariant with temperature.

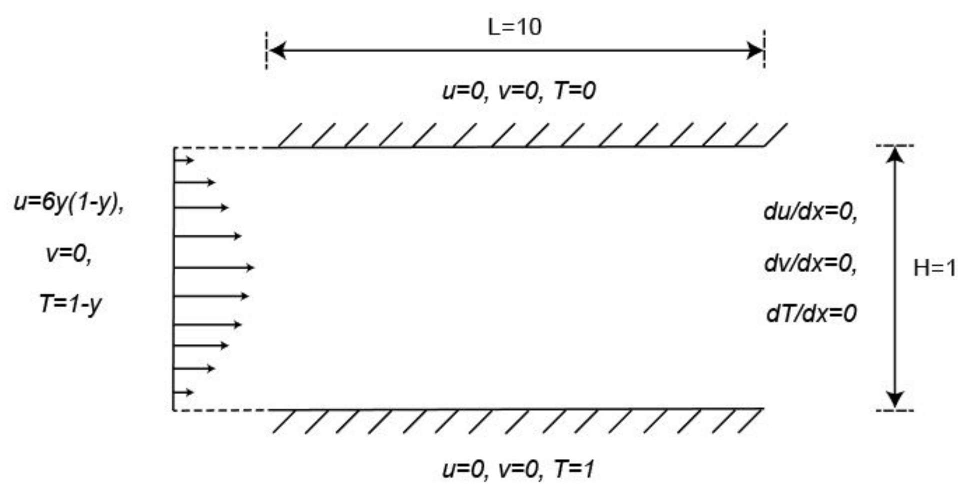

Figure 1: Poiseuille Benard Flow

Governing Equation

Under the assumption of constant-property Boussinesq fluid, in the absence of volumetric heating and neglecting the effects of viscous dissipation; the streamfunction, vorticity, and energy equations can be written in dimensionless form as:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) + \frac{Ra}{Re^2 Pr}\frac{\partial T}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{RePr}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

In the above equations space coordinates, velocities and time are normalized with respect to reference length L, to the average flow velocity u_{av}, and to a characteristic time $t_o = L/u_{av}$, respectively, while the dimensionless variable $T = (T - T_c)/(T_h - T_c)$ is referred to the defined "hot" and "cold" temperatures.

Spatial Discretization

The spatial discretization is carried out using Finite Difference approximations of the derivatives. Second order accurate approximations are defined at every point in the domain. At the boundaries, one sided second order approximations for the first and second order derivatives are defined. In the interior of the domain, central differencing is done for the first and second order derivatives.

At the boundaries,

$$\frac{\partial u}{\partial x}_i = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} \qquad or \qquad \frac{\partial u}{\partial x}_i = \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2}_i = \frac{2u_i - 5u_{i-1} + 4u_{i-2} - u_{i-3}}{\Delta x^2} \qquad or \qquad \frac{\partial^2 u}{\partial x^2}_i = \frac{2u_i - 5u_{i+1} + 4u_{i+2} - u_{i+3}}{\Delta x^2}$$

In the interior,

$$\frac{\partial u}{\partial x}_i = \frac{u_{i+1} + u_{i-1}}{2\Delta x} \qquad \frac{\partial^2 u}{\partial x^2}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

Temporal Discretization

Temporal discretization is carried out using a second-order semi-implicit time integration scheme. The time integration scheme is formed as a combination of Crank-Nicolson scheme acting on the diffusive terms and Adams-Bashforth scheme acting on the convective terms. Combination of an explicit second-order Adams-Bashforth scheme for the advection terms and an implicit Crank-Nicolson scheme for diffusion terms can be described as follows:

$$NL(\phi) = \frac{3}{2}NL(\phi^n) - \frac{1}{2}NL(\phi^{n-1})$$

$$D(\phi) = \frac{D(\phi) + D(\phi^n)}{2}$$

where $NL(\phi)$ and $D(\phi)$ represent,

$$NL(\phi) = u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} \qquad D(\phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

Vorticity and Temperature equations are treated according to this scheme and solved at every time step.

Boundary Conditions

Flow is assumed to be fully developed at the inlet and the top, bottom walls are subjected to no slip boundary conditions. Outlet boundary conditions are specially defined in correlation with stable open boundary conditions. A linear temperature profile is provided at the inlet and constant values are assigned to the top and bottom walls.

Streamfunction and Vorticity are derived from primitive variables and so their boundary conditions also need to be derived from that of the primitive variables. The walls act as streamlines in the flow and so ψ is taken to be constant along the walls. Using the velocity profile at the inlet we can obtain ψ profile at the inlet subject to values at the walls. At the walls the normal gradient of streamfunction is zero and that property is inherited in the boundary conditions of vorticity. The boundary conditions for ψ and ω are as follows:

At the inlet,

$$u = 6y(1-y), \qquad \psi = 3y^2 - 2y^3, \qquad \omega = 12y - 6$$

For the top and bottom walls $\psi = constant$ and is governed by the profile given above. In vorticity, we inculcate the normal gradient of ψ as zero condition and obtain the values as,

$$\frac{\partial \psi}{\partial y} = 0 \qquad implies \qquad \omega_n = -\frac{\partial^2 \psi}{\partial x^2}_{n-1}$$

At the top wall,

$$\psi = 1, \qquad \omega = -\frac{\partial^2 \psi}{\partial x^2}_{top,n-1}$$

At the bottom wall,

$$\psi = 0, \qquad \omega = -\frac{\partial^2 \psi}{\partial x^2}_{bottom,n-1}$$

The outlet boundary conditions for this transient problem are suggested to be used as "advective derivative conditions" by Comini et al. for vorticity and temperature. These conditions can be written as:

$$\frac{\partial \omega}{\partial t} + u_{av}\frac{\partial \omega}{\partial x} = 0$$

$$\frac{\partial T}{\partial t} + u_{av}\frac{\partial T}{\partial x} = 0$$

Streamfunction outflow conditions are derived as,

$$\frac{\partial \psi}{\partial x} = u_{av}\Delta t\left(\frac{\partial^2 \psi}{\partial x^2} + \omega\right)^{n-1} + \frac{\partial \psi}{\partial x}^{n-1}$$

So at the outlet considering $u_{av} = 1$,

$$\frac{\partial \psi}{\partial x} = \Delta t\left(\frac{\partial^2 \psi}{\partial x^2} + \omega\right)^{n-1} + \frac{\partial \psi}{\partial x}^{n-1}, \qquad \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial x} = 0$$

Temperature boundary conditions are set in order to facilitate easy handling of the values without loosing the physical significance.

At the inlet and walls

$$T_{inlet} = 1 - y, \qquad T_{top} = 0, \qquad T_{bottom} = 1$$

At the outlet,

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = 0$$

Results

The transient problem is solved for $Ra = 10^4$ and $Re = 10$ with time step of the order of $\Delta t = 10^-3$ necessary for ensuring proper solution without any wild oscillations contributing to divergence. Time marching is carried out till 50,000 time steps equivalent to 50 seconds of simulation time and the results are extracted from that run.

Streamline contours are plotted at regular time intervals during the solution to observe the periodic evolution of the streamlines in the domain as shown in Fig. 2 and 3. Velocity vectors are plotted intermediately for visualizing the flow better in terms of direction of the swirls as shown in Fig. 4 and 5. Temperature contours are also plotted to observe their periodicity as shown in Fig. 6.

The transient solution is said to have reached its dynamic steady state by observing the oscillations in variable values at a point. In this problem, a point in the center of the domain is tracked with time and the variable values are written in a file to enable plotting a time graph and visualize the oscillations as shown in Fig. 7, 8 and 9.

It was observed that using the fully developed boundary conditions at the outflow for vorticity and temperature leads to disturbances in the upstream direction near the outlet. These disturbances were confirmed by plotting the streamfunction contours over several time steps using fully developed boundary conditions at the outflow as shown in Fig. 10.

Qualitative Comparison

Comini el al. carried out the analysis of this problem using Finite Element Method and plotted streamfunction and temperature contours at regular time intervals to obey the periodic evolution. Their results can be shown as in Fig. 11

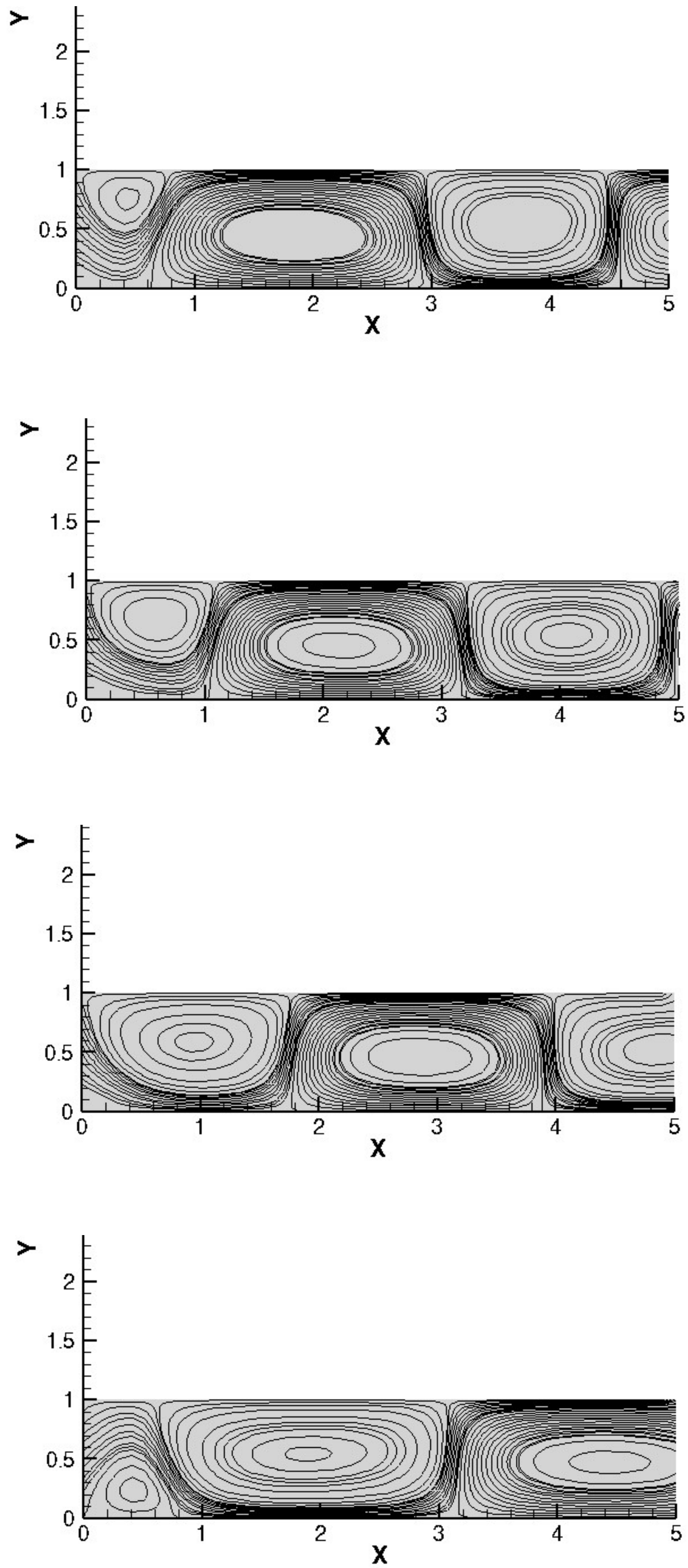

Figure 2: Comparison of streamfunction contours at various time steps

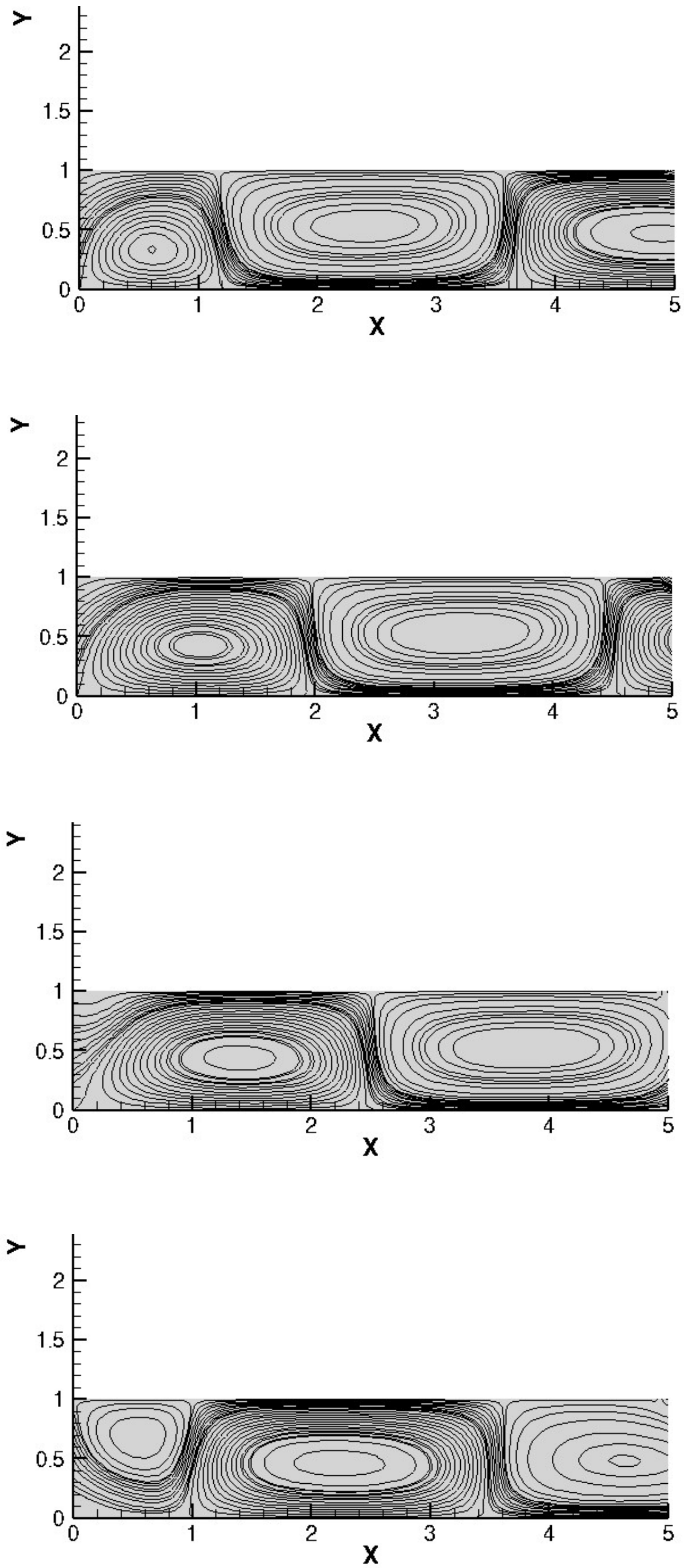

Figure 3: Comparison of streamfunction contours at various time steps

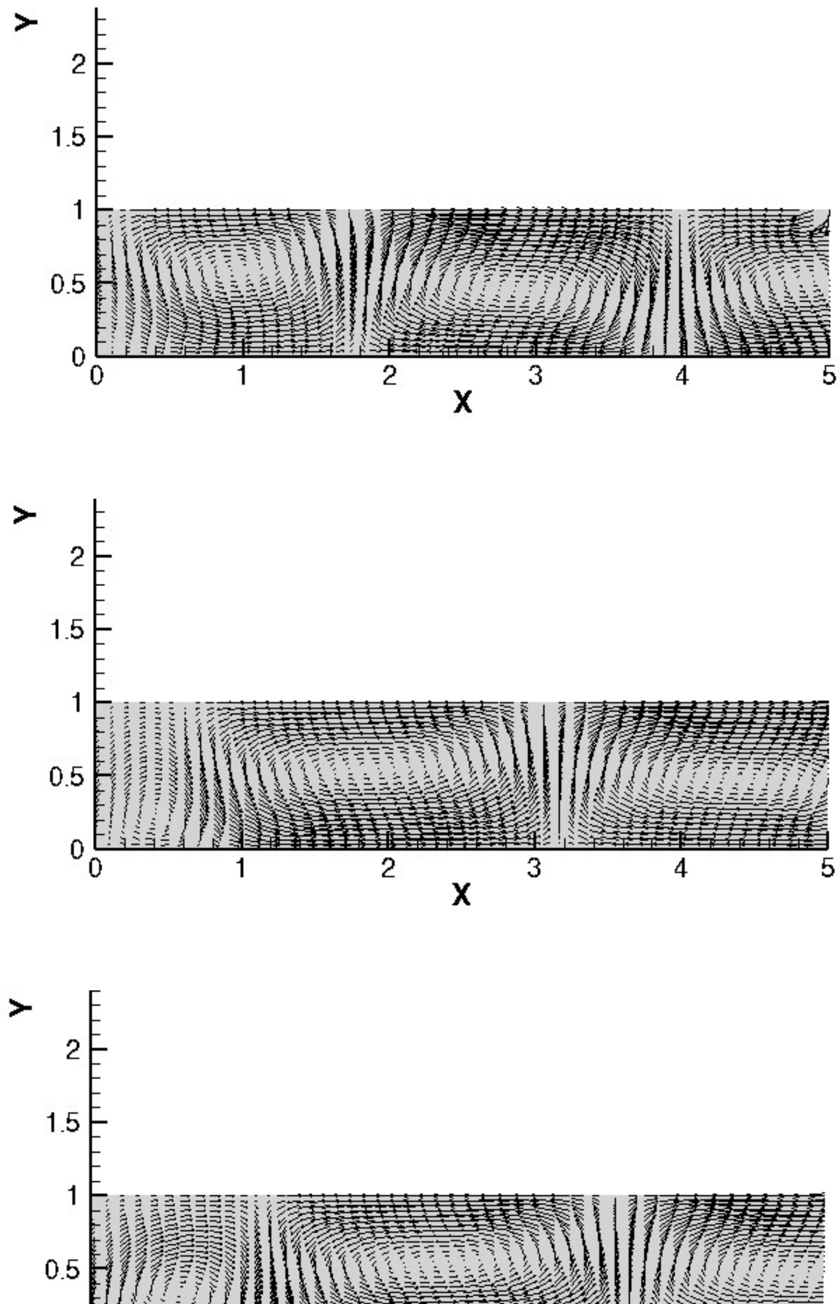

Figure 4: Comparison of Velocity vectors at different time steps

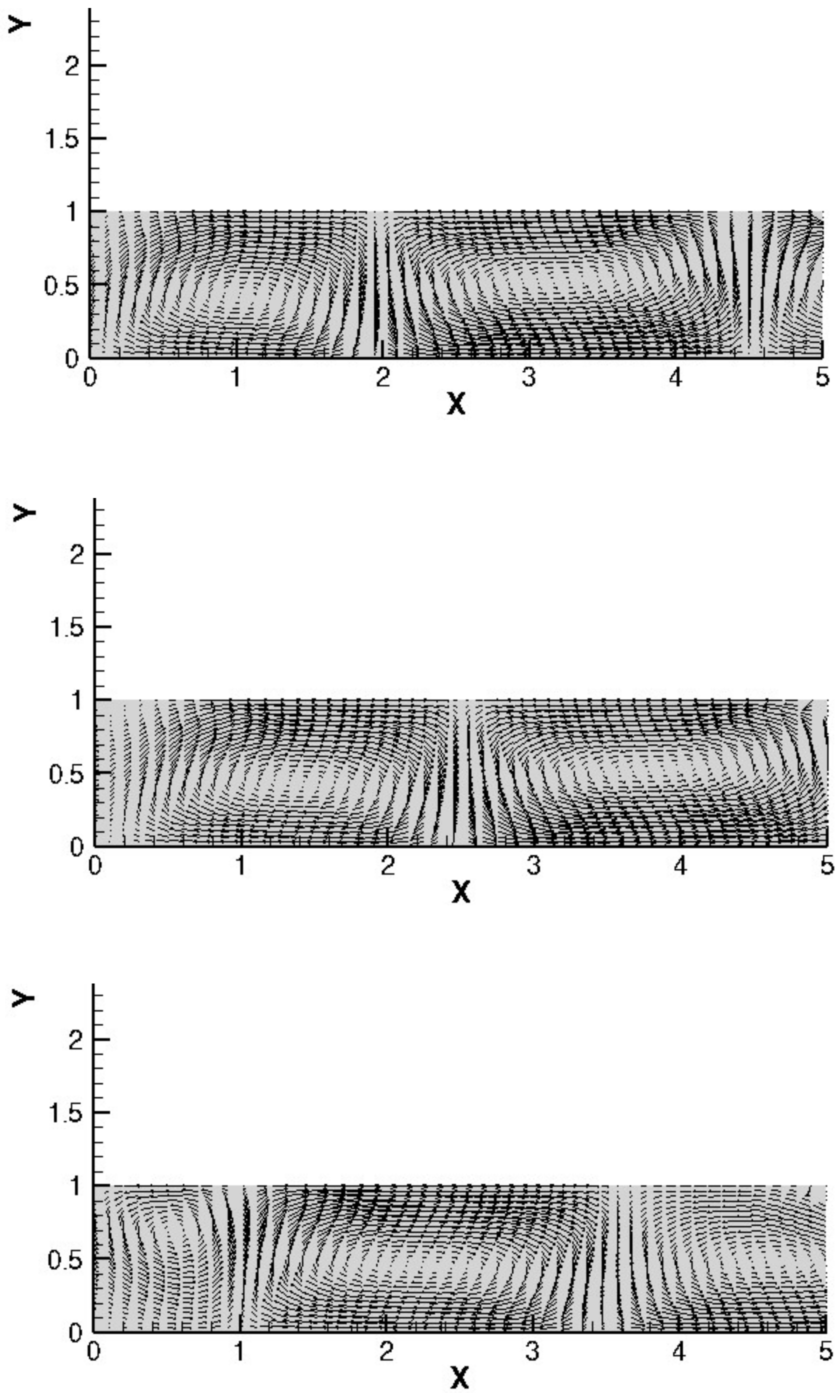

Figure 5: Comparison of Velocity vectors at different time steps

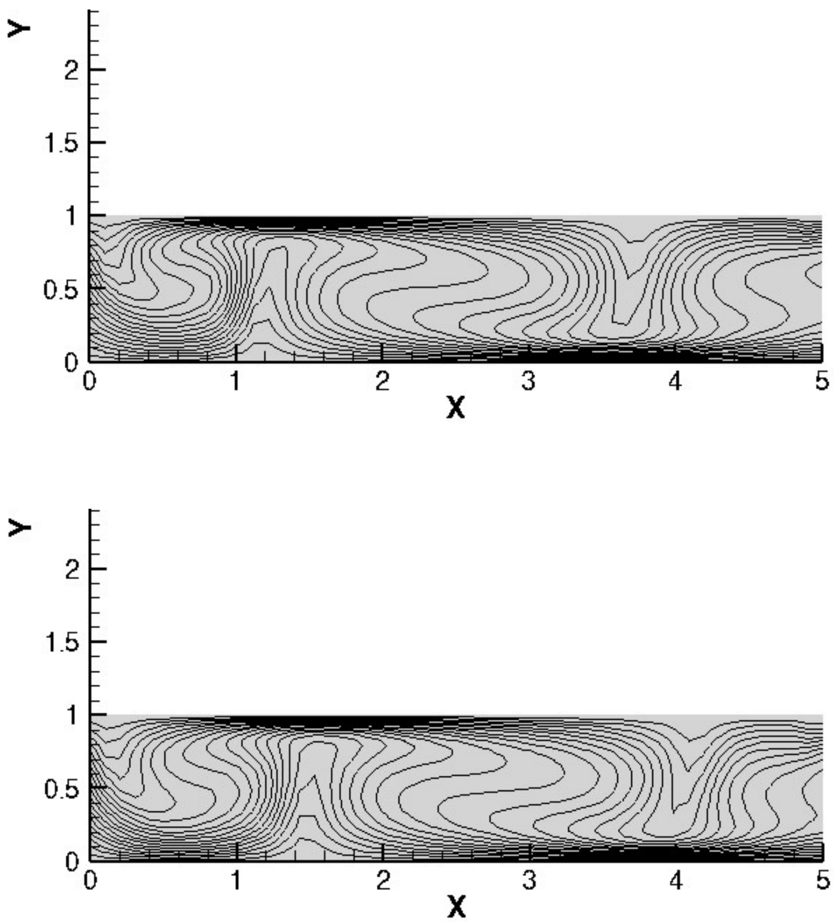

Figure 6: Comparison of Temperature Contours at different time steps

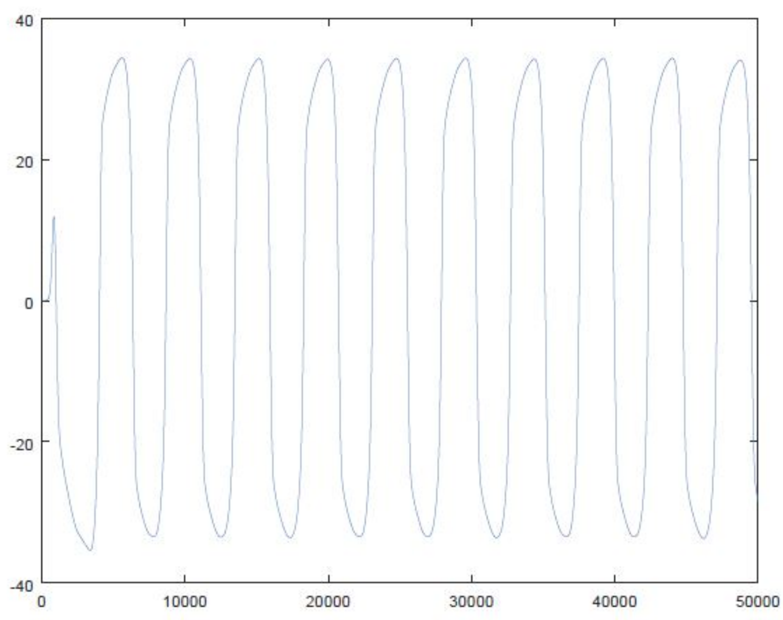

Figure 7: Periodicity in ω with time steps

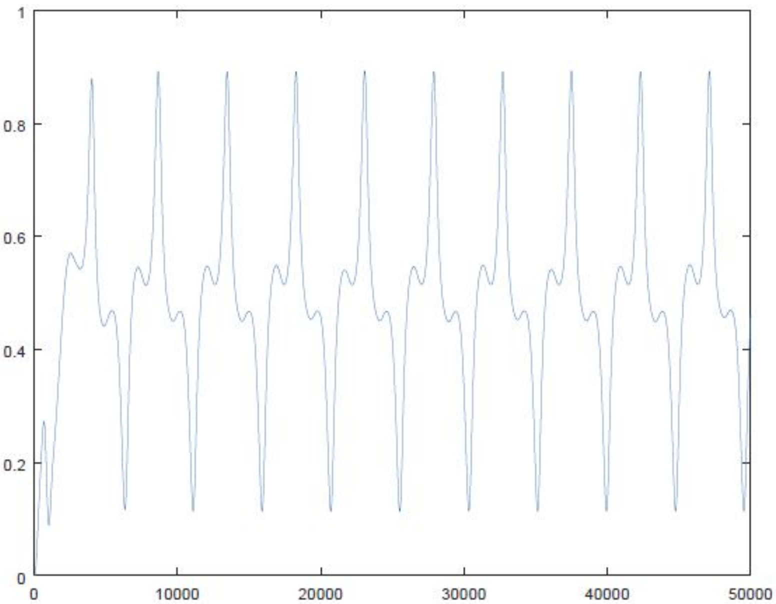

Figure 8: Periodicity in Temperature with time steps

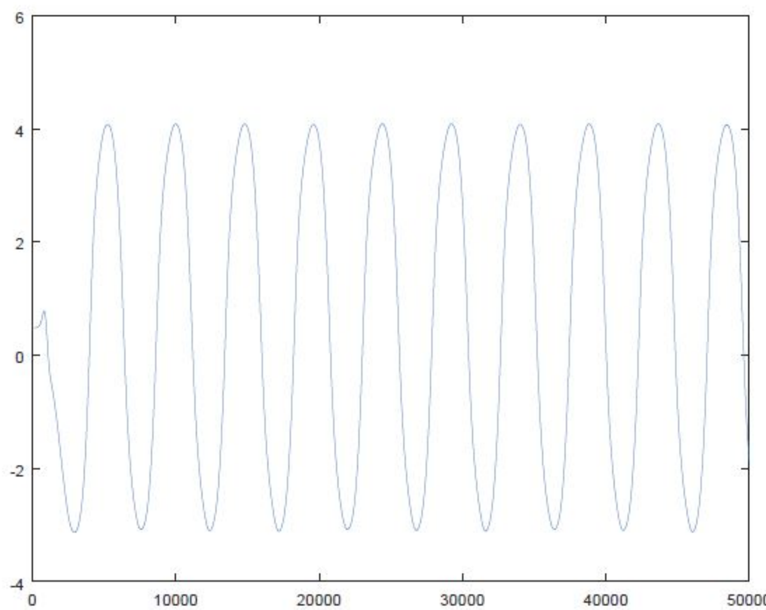

Figure 9: Periodicity in ψ with time steps

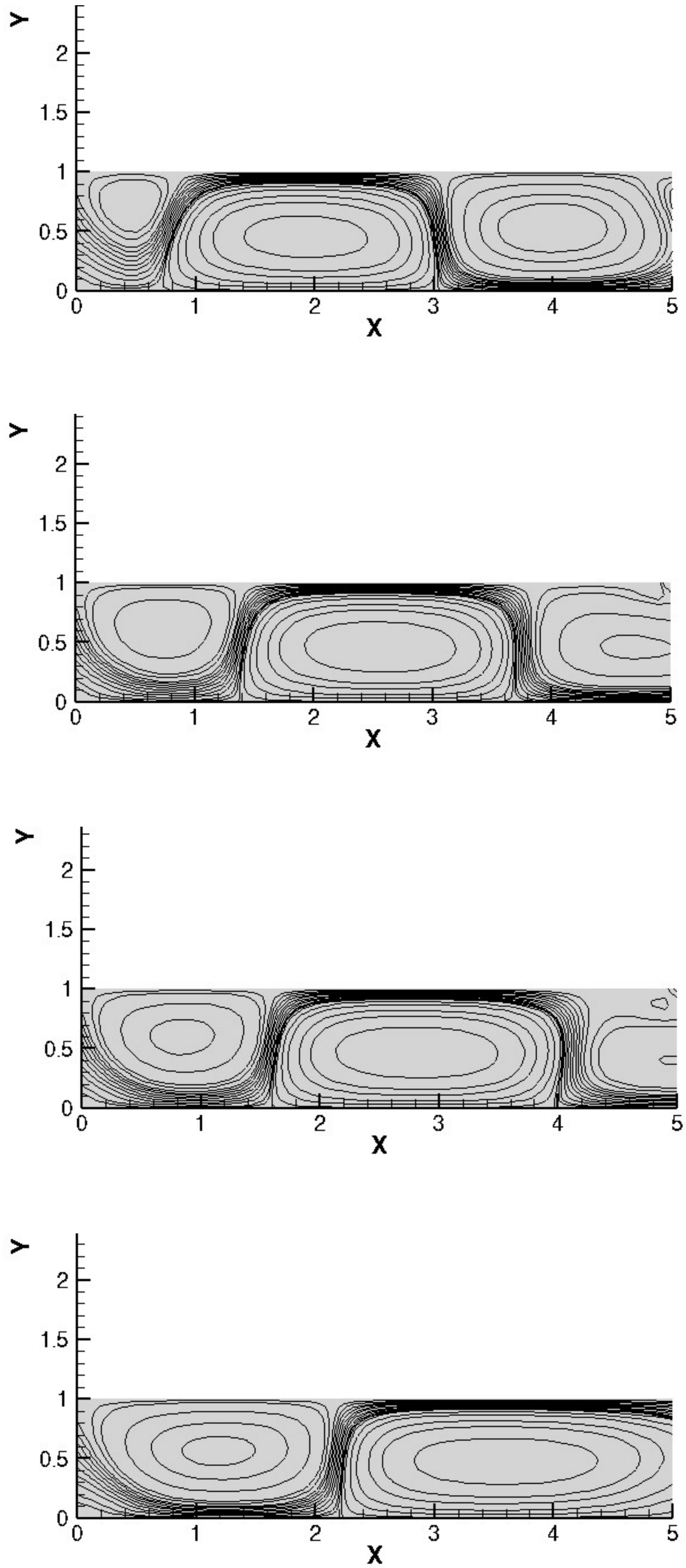

Figure 10: Smearing of streamfunction contours at various time steps using fully developed condition

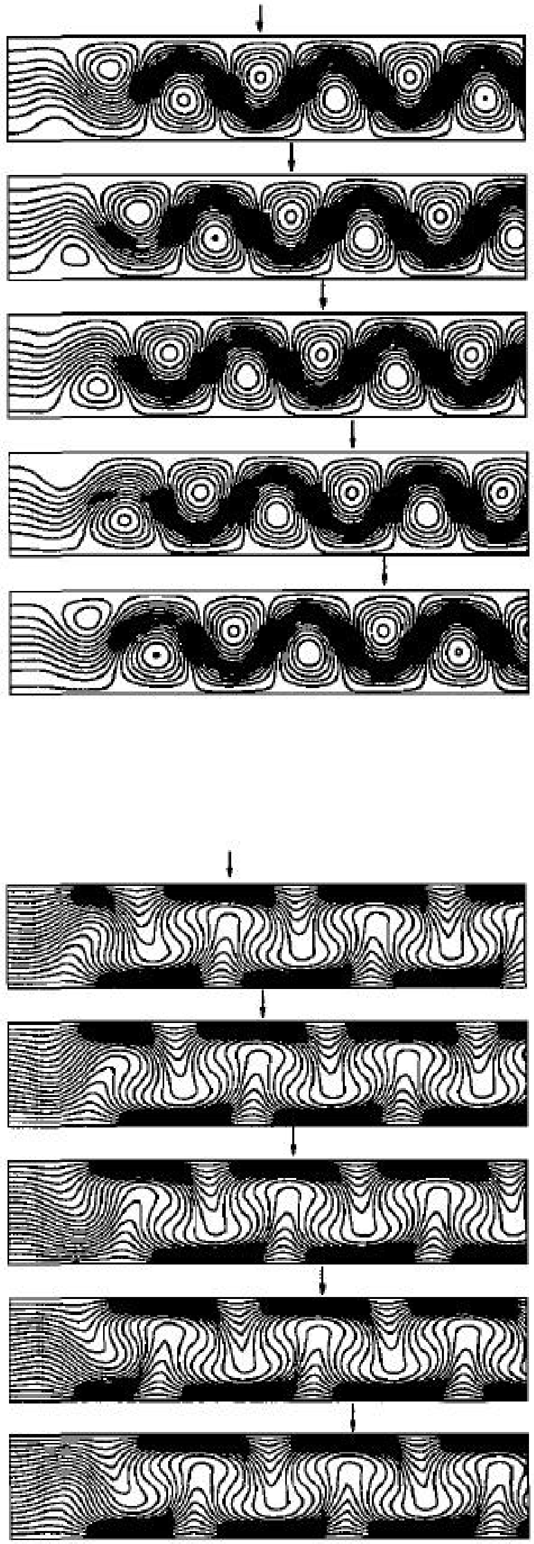

Figure 11: Streamfunction and Temperature contours from Comini et al.